**Project 2:** Compare the two concepts of matrix similarity and matrix congruence.

The definition of congruence can be found online (Matrix Congruence, 2021). Let A and B be two square matrices over a field F. We say that A is F-congruent to B if there exists an invertible matrix P∈Mn(F) such that PTAP = B. Congruence is an example of an equivalence relation. For two matrices to be congruent, they must be square and of the same dimension.

The following example is motivated by a statement found on a Mathematics website (Congruent and Similar Matrices, 2013) highlight the differences between a similar and congruent matrix. Consider the matrices:

1 1

0 1

1 0

0 1

A=

P=

P is an example of an invertible matrix that is not orthogonal. As mentioned in lecture, an orthogonal matrix, is given if the product of the matrix and the transpose of the matrix, is the identity matrix. Whereas we define A to be the identity matrix as it is only ‘similar’ to itself.

So, we have:

1 1

1 2

1 0

1 1

=

1 1

0 1

=

1 1

0 1

1 0

1 1

1 0

0 1

B= PTAP =

So, we’ve found matrices that are congruent but not similar.

J.J. Sylvester (1814-1987) was an English mathematician who contributed to many fundamental results in matrix theory. One of these include Sylvester’s criterion. That is if we let the matrices A and B be real symmetric, the leading principle minors are positive. As, the leading principal minors of A are 1,1 and the leading principal minors of B are 1, 1. The criterion has been satisfied.

Another contribution to matrix theory was Sylvester’s law of inertia (Sylvester’s law of inertia, 2022). This follows from the definition of diagonalization in the notes. That is: two symmetric n × n matrices A and B are congruent if and only if the diagonal representations for A and B have the same rank, index, and signature. Applying this to the example above matrices A and B have a rank of 2, index 1 and signature of 0. Thus, this condition is satisfied. Once again, showing that the two matrices are congruent.

A picture containing text, clock

Description automatically generatedOne point of commonality between the concepts of similarity and congruence is that they are both equivalence relations. Of similar matrices, we have the following definition (Matrix Similarity, 2022). Let A and B be two n-n matrices over a field F. We say that A is F-similar to B if there exists an invertible matrix P∈Mn(F) such that . From the definition of congruency, we have that A and B are equivalent if and only if they represent the same bilinear form. Comparatively, similarity is an equivalence relation as the properties of injectivity, surjectivity and reflexivity are satisfied.

Similar matrices represent the same linear map under two (possibly) different bases. However, neither the new basis nor the original need to be orthogonal. Whereas congruent matrices represent the same bilinear form with respect to different bases. Therefore, linear transformations are related to similar matrices, in the same way that bilinear forms are related to congruent matrices.

**Bibliography**

Matrix Congruence (2021) Available at: <https://en.wikipedia.org/wiki/Matrix_congruence#:~:text=Matrix%20congruence%20arises%20when%20considering,with%20respect%20to%20different%20bases>. (Accessed: 26 November 2022).

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